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2006 J. Phys. A: Math. Gen. 39 6039

(http://iopscience.iop.org/0305-4470/39/20/027)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 6039–6044 doi:10.1088/0305-4470/39/20/027

Yang–Mills theory for bundle gerbes

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Received 5 February 2006
Published 3 May 2006
Online at stacks.iop.org/JPhysA/39/6039

Abstract

Given a bundle gerbe with connection on an oriented Riemannian manifold of dimension at least equal to 3, we formulate and study the associated Yang–Mills equations. When the Riemannian manifold is compact and oriented, we prove the existence of instanton solutions to the equations and also determine the moduli space of instantons, thus giving a complete analysis in this case. We also discuss duality in this context.

PACS numbers: 02.40.Ma, 02.40.Vh, 03.50.De
Mathematics Subject Classification: 70S15, 81T13

Introduction

The Yang–Mills equations for a line bundle on a Riemannian manifold, otherwise known as Abelian Yang–Mills equations, are an elegant reformulation of Maxwell's equations for electromagnetism on a general Riemannian manifold X. It has been extensively studied in physics, and a nice account of it from a mathematician's perspective can be found in [1]. Just as line bundles are classified up to isomorphism by $H^2(X, \mathbb{Z})$, it is known that bundle gerbes, which were invented in [5], are classified up to stable isomorphism by $H^3(X, \mathbb{Z})$ [6]. It is natural to ask whether there is an analogue of Yang–Mills theory for bundle gerbes on Riemannian manifolds?

In section 1, we formulate and study the analogue of the Yang–Mills equations for bundle gerbes with connection on Riemannian manifolds X of dimension at least equal to 3. When X is a compact oriented Riemannian manifold, we prove in section 2 the existence of instanton solutions to these equations and in corollary 2 we establish that the moduli space of instanton solutions is isomorphic to a torus $\mathbb{T}^{b^2(X)}$ of dimension equal to the second Betti number of X, thus giving a complete analysis in this case. Moreover, if we allow the bundle gerbe connection on the bundle gerbe to vary and if $\mathcal{A}$ denotes the affine space of all bundle gerbe connections on the given bundle gerbe, then in corollary 3 we deduce that the moduli space of instantons is a fibre bundle over $\mathcal{A}$ with fibres isomorphic to the torus $\mathbb{T}^{b^2(X)}$. We also discuss duality in our context, and in corollary 1 we give a new geometric interpretation of the de Rham cohomology group $H^2(X, \mathbb{R})$.

In future research, we plan to generalize our results to n-bundle gerbes, $n > 1$, and also to non-Abelian bundle gerbes. We mention that there have been other approaches to 'higher' versions of Yang–Mills theory [2, 3 and references there in], which use (higher) category theory and so are rather different from our pedestrian geometric approach in this paper.

1. Preliminaries

1.1. Bundle gerbes and bundle gerbe connections

The material in this section is a very brief review of bundle gerbes. For details, we refer the reader to [5, 6].

Let $(M, \mathcal{L})$ be a bundle gerbe over compact oriented manifold X. That is, $M \stackrel{\pi}{\to} X$ is a submersion and $\mathcal{L} \to M^{[2]}$ is a primitive line bundle over the fibred product $M^{[2]} = \Delta^*(M \times M)$ where $\Delta : X \to X \times X$ is the diagonal map, $x \mapsto (x, x)$. We recall that a primitive line bundle is one that comes equipped with isomorphisms,

$$\mathcal{L}_{(x,y)} \otimes \mathcal{L}_{(y,z)} \cong \mathcal{L}_{(x,z)}, \tag{1}$$

for all $x, y, z \in M$, called the bundle gerbe product.

Recall from [5] that for any fixed $p \geqslant 0$, we have an *exact* complex

$$\Omega^p(X) \stackrel{\pi^*}{\to} \Omega^p(M) \stackrel{\delta}{\to} \Omega^p(M^{[2]}) \stackrel{\delta}{\to} \cdots. \tag{2}$$

Here $\delta\colon \Omega^p(M^{[q]}) \to \Omega^p(M^{[q+1]})$ is the alternating sum of pull-backs $\sum_{j=1}^{q+1}(-1)^j \pi_j^*$ of projections, where π_i is the projection map which omits the ith point in the fibre product, and $M^{[q]}$ denotes the qth fibred product.

We will always assume that $M^{[2]}$ admits partitions of unity, in which case $\mathcal{L} \to M^{[2]}$ admits connections. It is then shown in [5] that $\mathcal{L}$ admits *bundle gerbe connections* that is connections respecting the bundle gerbe product. A bundle gerbe connection ∇ has curvature F_∇ satisfying $\delta(F_\nabla) = 0$ and hence from the exactness of the fundamental complex (2) there exists a 2-form f on M, satisfying the 'descent equation'

$$F_\nabla = \pi_1^*(f) - \pi_2^*(f).$$

Such an f is called a *curving* for the connection ∇. Let $\mathcal{C} = \mathcal{C}(M, \mathcal{L}, \nabla)$ denote the space of all curvings for the connection ∇. Then $\mathcal{C}$ is an affine space associated with the vector space $\Omega^2(X)$. To see this, observe that if $f_1, f_2 \in \mathcal{C}$, then $0 = \pi_1^*(f_1 - f_2) - \pi_2^*(f_1 - f_2) = \delta(f_1 - f_2)$. Therefore, by the exactness of the fundamental complex, $f_1 - f_2 = \delta(\lambda) = \pi^*(\lambda)$ where $\lambda \in \Omega^2(X)$. Note that $\mathcal{C}$ actually only depends on the curvature F_∇. Given a choice of curving, we then have that $\delta(df) = d\delta(f) = dF_\nabla = 0$ so that by the exactness of the fundamental complex (2) we can find a 3-form $H(f)$ on X, such that $df = \delta(H(f)) = \pi^*(H(f))$. Moreover, $H(f)$ is closed as $\pi^*(dH(f)) = ddf = 0$. In [5], it is shown that $H(f)/2\pi\mathrm{i}$ has integral periods is a de Rham representative for the Dixmier–Douady class. Here $H(f)$ is called the *3-curvature* of the connection and curving (∇, f). It is shown in [5] that the cohomology class $[H(f)/2\pi\mathrm{i}] \in H^3(X, \mathbb{Z})$ is independent of the choice of curving f.

2. Yang–Mills functional for bundle gerbes, critical points and duality

The configuration space for the Yang–Mills functional for the bundle gerbe $(M, \mathcal{L})$ is defined as the space of all curvings $\mathcal{C} = \mathcal{C}(M, \mathcal{L}, \nabla)$ for the bundle gerbe connection ∇ on $(M, \mathcal{L})$. We define the Yang–Mills functional for the bundle gerbe $(M, \mathcal{L})$ with bundle gerbe connection ∇ as

$$\begin{aligned} &\mathrm{YM}\colon \quad \mathcal{C} \to \mathbb{R}, \\ &\mathrm{YM}(f) = \int_X H(f) \wedge *H(f), \end{aligned} \tag{3}$$

where * denotes the Hodge star operator, with respect to the given Riemannian metric on X. Note that the compactness of X is used here to ensure that the integral is finite. The Euler–Lagrange equations are then derived in the standard way,

$$\mathrm{YM}(f + \varepsilon h) - \mathrm{YM}(f) = 2\varepsilon \int_X H(h) \wedge *H(f) + O(\varepsilon^2), \tag{4}$$

for all $h \in \mathcal{C}$. Therefore, $d^* H(f) = 0$, where d^* denotes the formal adjoint of the de Rham operator d. But we always have $dH(f) = 0$, so we conclude that the critical points of the Yang–Mills functional YM for bundle gerbes satisfy the following Yang–Mills equations for bundle gerbes:

$$dH(f) = 0, \qquad d^* H(f) = 0. \tag{5}$$

Note that these equations continue to make sense for noncompact Riemannian manifolds.

2.1. Duality

Note that since $d^* = \pm * d *$ and $*^2 = \pm 1$ or $\pm \mathrm{i}$, we see that the Yang–Mills equations for bundle gerbes (5) are invariant under the transformation $H \mapsto *H$, which is the analogue of the electromagnetic duality for Abelian Yang–Mills.

2.2. Gauge group for bundle gerbes and its action

Define the gauge group $\mathcal{G} = \mathcal{G}(X, \mathcal{L})$ to be $C^\infty(X, \mathrm{PU})$, where $\mathrm{PU} = U/\mathbb{T}$ is the projective unitary group on an infinite-dimensional, separable Hilbert space. Then $\mathcal{G}$ acts on $\mathcal{C}$ via

$$\gamma \cdot f = f + \pi^* \gamma^* (\omega),$$

where $\omega \in \Omega^2(\mathrm{PU})$ is a closed 2-form such that $[\omega] \in H^2(\mathrm{PU}, \mathbb{Z})$ is the generator, $\gamma \in \mathcal{G}$ and $f \in \mathcal{C}$. In fact, we will make a particular choice of ω, which is primitive. More precisely, recall that the line bundle L associated with the central extension $\mathbb{T} \to U \to \mathrm{PU}$ is primitive in the sense that there are canonical isomorphisms $L_g \otimes L_h \cong L_{gh}$ for all $g, h \in \mathrm{PU}$, and there is a connection ∇, on the line bundle L, called a primitive connection, which is compatible with these isomorphisms. If ω is the curvature of such a primitive connection ∇, then we see that $\omega_g + \omega_h = \omega_{gh}$ for all $g, h \in \mathrm{PU}$, and ω is said to be a primitive closed 2-form on PU. Suppose now $\eta, \gamma \in \mathcal{G}$ and $x \in X$. Then, denoting the pointwise product of η and γ as $\eta\gamma$, we have

$$\begin{aligned} ((\eta\gamma)^* \omega)_x &= \omega_{\eta(x)\gamma(x)} \\ &= \omega_{\eta(x)} + \omega_{\gamma(x)} \\ &= (\eta^* \omega)_x + (\gamma^* \omega)_x. \end{aligned}$$

Thus, $(\eta\gamma)^* \omega = \gamma^* \omega + \eta^* \omega$. The action of the gauge group $\mathcal{G}$ on $\mathcal{C}$ can now be seen to be well defined, namely

$$(\eta\gamma) \cdot f = (f + \pi^* \gamma^* \omega) + \pi^* \eta^* \omega = \eta \cdot (\gamma \cdot f)$$

for all $f \in \mathcal{C}$.

Observe that $d(\gamma \cdot f) = df$ for all $\gamma \in \mathcal{G}$ and $f \in \mathcal{C}$, i.e.

$$H(\gamma \cdot f) = H(f).$$

In particular, YM $(\gamma \cdot f) =$ YM (f), i.e.

$$\mathrm{YM}\colon \mathcal{C}/\mathcal{G} \to \mathbb{R},$$

is a well-defined Morse (quadratic) function on $\mathcal{C}/\mathcal{G}$.

Remark 1. Instead of the group PU, we could have chosen any other differentiable group G such that G is an Eilenberg–Maclane space $K(\mathbb{Z}, 2) = BU(1)$. For instance, G can be even chosen to be an Abelian group which is a differentiable space, cf [4], which is a weak form of smooth structure for infinite-dimensional spaces.

3. Existence of (instanton) solutions and moduli space

3.1. Existence

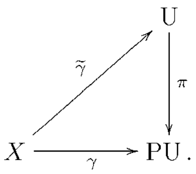
Since X is a compact oriented manifold, by the Hodge theorem, cf [8], which states that every cohomology class on a compact oriented manifold has a unique harmonic representative, there is a unique 3-curvature (instanton) solution to the Yang–Mills equations for bundle gerbes (5).

3.2. Moduli space

Our next goal is to determine the space of all (instanton) solutions to these equations and also the moduli space of gauge equivalent (instanton) solutions to the equations. That is, we want to analyse the set of all $f \in \mathcal{C}$ such that $H(f) = H(f_0)$ for some fixed $f_0 \in \mathcal{C}$. That is, $d(f - f_0) = 0$. Since we always have $f - f_0 \in \Omega^2(X)$, we see that the difference $f - f_0$ is a closed 2-form, $f - f_0 \in Z^2(X)$, where $Z^2(X)$ denotes the vector space of all closed 2-forms on X. Now the induced action of $\mathcal{G}$ on $Z^2(X)$ is $\gamma \cdot \xi = \xi + \gamma^*\omega$, where $\gamma \in \mathcal{G}$, $\xi \in Z^2(X)$ and ω is as in section 2.2. Consider the subgroup $\mathcal{G}_0$ of $\mathcal{G}$ consisting of all smooth maps $X \mapsto$ PU that are null homotopic. Then we have

Proposition 1. *Let $B^2(X)$ denote the space of all exact 2-forms on X. For all $\gamma \in \mathcal{G}_0$, the pull-back $\gamma^*\omega \in B^2(X)$, where ω is a primitive closed 2-form on* PU *such that $[\omega] \in H^2(\mathrm{PU}, \mathbb{Z}) \cong \mathbb{Z}$ is the generator. Moreover, for any exact 2-form $F \in B^2(X)$, there is a $\gamma \in \mathcal{G}_0$ such that $F = \gamma^*\omega$.*

Proof. For $\gamma \in \mathcal{G}_0$, by the homotopy invariance of de Rham cohomology, $[\gamma^*\omega] = 0$, therefore $\gamma^*\omega \in B^2(X)$. An alternate proof is given as follows. Note that any $\gamma \in \mathcal{G}_0$ lifts to a smooth map $\widetilde{\gamma} : X \to U$ where U denotes the unitary group of the given Hilbert space. That is, we have the commutative diagram



(6)

Since U is contractible, there is a 1-form $\Lambda \in \Omega^1(U)$ such that $\pi^*(\omega) = d\Lambda$. By the commutativity of diagram (6), we have $\gamma^*(\omega) = d\widetilde{\gamma}^*(\Lambda)$, that is, $\gamma^*(\omega) \in B^2(X)$.

Conversely, given $F \in B^2(X)$, we need to show that there is a smooth map $h : X \to U$ such that $F = dh^*(\Lambda)$ where $\Lambda \in \Omega^1(U)$ is such that $\pi^*(\omega) = d\Lambda$. Then the composition $\gamma = \pi \circ h \in \mathcal{G}_0$ has the property that $F = \gamma^*\omega$, as desired. But, by the theory of universal connections, cf [7], there is a smooth map $h_0 : X \to S$, where S denotes the unit sphere in the given Hilbert space such that $F = dh_0{}^*(\Lambda_0)$ where $\Lambda_0 \in \Omega^1(S)$ is the universal connection, having the property that $\bar{\pi}^*(\omega_0) = d\Lambda_0$, where $\omega_0 \in \Omega^2(P)$ denotes the universal curvature 2-form on the projectivized Hilbert space P, where $\bar{\pi} : S \to P$ is the circle bundle. Pick a point $x_0 \in S$. Then there is a map $q : U \to S$, which sends a unitary operator T to the point $T(x_0)$ on the unit sphere. Since S and U are contractible, there is a lift $h : X \to U$ of h_0. That is, we have the commutative diagram

$$\begin{array}{ccc} & & \mathrm{U} \\ & \overset{h}{\nearrow} & \downarrow q \\ X & \xrightarrow[h_0]{} & \mathrm{S}. \end{array} \tag{7}$$

Define $\Lambda = q^*(\Lambda_0)$. Then by the commutativity of diagram (7), we see that $F = dh^*(\Lambda)$ as desired. The particular choice of ω that we make is $\omega = \bar{q}^*(\omega_0)$, where $\bar{q}$: PU $\to P$ is the map induced by q, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{U} & \xrightarrow{q} & \mathrm{S} \\ \downarrow \pi & & \downarrow \bar{\pi} \\ \mathrm{PU} & \xrightarrow{\bar{q}} & \mathrm{P}. \end{array} \tag{8}$$

□

The following corollary can be viewed as giving a new geometric interpretation of the cohomology group $H^2(X, \mathbb{R})$.

Corollary 1. *Let $\mathcal{M}^0_\nabla$ denote the moduli space of null homotopic gauge equivalent (instanton) solutions to the Yang–Mills equations for the bundle gerbe $(M, \mathcal{L})$ with connection ∇ over a compact oriented manifold X. Then $\mathcal{M}^0_\nabla$ is isomorphic to the cohomology group $H^2(X, \mathbb{R})$.*

Proof. By definition, $\mathcal{M}^0_\nabla = Z^2(X)/\mathcal{G}_0$. By proposition 1, the quotient $Z^2(X)/\mathcal{G}_0 = H^2(X, \mathbb{R})$, proving the corollary. □

Corollary 2. *Let $\mathcal{M}_\nabla$ denote the moduli space of gauge equivalent (instanton) solutions to the Yang–Mills equations for the bundle gerbe $(M, \mathcal{L})$ with connection ∇ over a compact oriented manifold X. Then $\mathcal{M}_\nabla$ is diffeomorphic to the torus $\mathbb{T}^{b^2(X)}$ of dimension equal to the second Betti number of X.*

Proof. By proposition 1, the quotient $Z^2(X)/\mathcal{G}_0 = H^2(X, \mathbb{R})$. Also, since PU is an Eilenberg–Maclane space $K(\mathbb{Z}, 2)$, we see that the group of components of the gauge group is $\pi_0(\mathcal{G}) = \mathcal{G}/\mathcal{G}_0 = [X, \mathrm{PU}] = H^2(X, \mathbb{Z})$. □

Therefore,

$$\begin{aligned} \mathcal{M}_\nabla &= Z^2(X)/\mathcal{G}, \\ &= (Z^2(X)/\mathcal{G}_0)/(\mathcal{G}/\mathcal{G}_0), \\ &= H^2(X, \mathbb{R})/\pi_0(\mathcal{G}), \\ &= H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}), \\ &\cong \mathbb{T}^{b^2(X)}. \end{aligned} \tag{9}$$

It has been shown in [5] that the space $\mathcal{A}$ of all bundle gerbe connections on the bundle gerbe $(M, \mathcal{L})$ over X is an affine space associated with the vector space $\Omega^1(M)/\pi^*(\Omega^1(X))$. Therefore, we have the following corollary.

Corollary 3. *Let $\mathcal{M} = \bigcup_{\nabla\in\mathcal{A}} \mathcal{M}_\nabla$ denote the moduli space of gauge equivalent (instanton) solutions to the Yang–Mills equations for the bundle gerbe $(M, \mathcal{L})$ over a compact oriented manifold X. Then $\mathcal{M}$ is diffeomorphic to a torus bundle over the affine space $\mathcal{A}$ with fibre isomorphic to the torus $\mathbb{T}^{b^2(X)}$ of dimension equal to the second Betti number of X.*

Acknowledgments

VM thanks Michael Murray for a conversation about universal connections. VM acknowledges support from the Australian Research Council and DR acknowledges the receipt of an Australian Postgraduate Award.

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